OPTIMAL CONTROLLABILITY OF NONLINEAR INFINITELY SPACE OF NEUTRAL DIFFERENTIAL SYSTEMS WITH DISTRIBUTED DELAYS IN THE CONTROL

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ABSTRACT

In this paper, the nonlinear system (1.1) was linearized and the mild solution of the system was obtained using the Unsymmetric Fubuni Theorem. The necessary and sufficient conditions for the existence of optimal control were established. We also, made use of the integration technique as in Klamka (1976) to establish results. The reachable set, attainable set and target set upon which our studies hinged were extracted from the mild solution.

Keywords: Optimality Control, Unsymmetric Fubini Theorem, Linearization, Neutral System, Set Functions.

1.0 Introduction

There are several papers which appeared on the controllability of nonlinear systems in infinite dimensional spaces.Balachandran K. and Anandhi E.R. (2003) discussed the controllability of neutral functional integro-differential systems in abstract phase space, with the help of Schander fixed point theorem. Fu (2004) studied the same problem in abstract phase space for neutral functional differential systems with unbounded delay by using the Sado vskii fixed point theorem.
Onwuatu, J.U.(1984), discussed the problem for nonlinear systems of neutral functional differential equations with limited controls.
However, the systems with delays in both the state and control, investigation into their relative controllability are still attracting attention and interest.
Optimality conditions for the relative controllability of nonlinear infinitely space of Neutral differential systems with distributed delays in the control is yet to be reported, though there are studies in the optimal controllability of ordinary and functional differential systems.
From the following studies, Artstein, Z. and Tradma G. (1982), Balachandran, K. and J.P. Dauer (1989), Chukwu, E.N. (2001), we gain clarity of the meaning and full understanding of the conceptual frame work of optimal controllability.
In this work, we shall consider the Nonlinear Infinitely Space of Neutral Differential Systems with Distributed Delays in the Control of the form:

$$\frac{d}{dt}\left[D\left(t,x_t\right)\right] = L\left(t,x_t\right)x_t + \int_{-\infty}^{0} A(t,\theta)x(t+\theta)d\theta + \int_{-h}^{0} d_\theta H(t,\theta)u(t+\theta) + f(t,x_t) \quad \ldots\ldots \quad (1.1)$$

$x_{t_0} = \phi$

With the main objective of investigating for the existence of an optimal control of the system (1.1).

1. Notation and Preliminaries

Let n be a positive integer and $E = (-\infty,\infty)$ be the real line.

Denote by E^n the space of real $n-$tuples called the Euclidean space with norm *denoted by* $|.|$.

If $J = [t_0, t_1]$ *is any interval of* E, L_2 *is the Lebesque space of square integrable functions from* J *to* E^n *written in full as* $L_2([t_0, t_1], E^n)$.

Let $h > 0$ *be positive real number and let* $C([-h, 0], E^n)$ *be the of Banach space of continuous functions with the norm of uniform convergence defined by*

$$\|\phi\| = \sup \phi(s);\ [-h, 0], \phi \in C([-h, 0], E^n)$$

If x *is a function from* $[-h, \infty]$ *to* E^n, *then* x_t *is a function defined on the delay interval* $[-h, 0]$ *given as*

$$x_t(s) = x(t + s); s \in [-h, 0], t \in [0, \infty).$$

Consider the nonlinear infinite neutral system

$$\frac{d}{dt}[D(t, x_t)] = L(t, x_t)x_t + \int_0^\infty A(t, \theta)\, x(t + \theta)d\theta$$

$$+ \int_{-h}^{0} [d_\theta\, H(t, \theta)]u(t + \theta) + f(t, x_t) \ \ldots\ldots (2.1)$$

$(Circularity\ of\ the\ function\ from\ -\infty\ to\ 0\ and\ from\ 0\ to\ \infty).$

Where

$$L(t, x_t) = \sum_{k=0}^{\infty} A_k\, x(t - w_k) + \int_{-\infty}^{0} A\,(t, \theta)x(t + \theta)d\theta.$$

$$L(t, x_t)x_t = \int_{-h}^{0} d_\theta\, \eta\big(t, s, x(t + s)\big)x(t + \theta)$$

$$\eta(t, s, \phi, \psi) \geq 0, for\ s \geq 0\ and\ \phi, \psi \in C.$$

$$\eta(t, s, \phi, \psi) exists\ for\ t < -h$$

$\eta(t, s, \phi, \psi)$ *is a continuous matrix function of bounded variation in* $s \in [-h, 0]$,

$$vary\ \eta(t) \leq m(t), \qquad m(t) \in L_1.$$

L_1 *is the space of integrable functions.*

Let Ω *be an open subset of* ExC *and* D, L *be bounded linear operators defined on* ExC *into* E^n

$$|L(t, x_t)x_t| \leq m(t)\|x_t\|, \qquad for\ all\ t \in E, \psi(t) \in C$$

$$D(t, x_t) = x(t)g\,(t, x_t),$$

$$Where\ g(t, x_t) = \sum_{n-o}^{\infty} A_n\,(t)\phi(t - w_n(t) + \int_{-h}^{0} A\,(t, s)\phi(s)ds = \int_{-h}^{0} d_\theta\, H(t, \theta)\phi(\theta)$$

$$Where, 0 \leq w_n \leq h, and \left|\int_{-h}^{0} d_\theta\, H(t,\theta)\phi(\theta)\right| \leq h(\theta)\|\phi\|.$$

$D(t,x_t)$ *is non − atomic at zero (that is* $D(t,x_t)$ *is differentiable and integrable at zero).*

$$\int_{-h}^{0} A\,(t,s)ds \;+ \sum_{n-1}^{\infty} |A(t)| \geq \;\delta(\varepsilon), for\ all\ t, where\ \delta(\varepsilon) \to 0,$$

f *is continuous and satisfies other smoothness conditions.*

1.1. Linearization

We can linearize the system (2.1) as in **Chukwu, E. N. (1992)** by setting $x_t = z\ in\ L$; a specified function insider the function $L(t,x_t)x_t$ to have $L(t,z)x_t$ without loss of generality. Thus system (2.1) becomes

$$\frac{d}{dt}[D(t,x_t)] = \;L(t,z)x_t \;+ \int_{0}^{\infty} A(t,\theta)\,x(t+\theta)d\theta$$

$$+\int_{-h}^{0} d_\theta\, H(t,\theta)]u(t+\theta) \;\;+\; f(t,x_t) \;\;\ldots..(2.2)$$

Evidently,

$$L(t,z)x_t \;\; = \sum_{k=0}^{\infty} A_k\, x(t-w_k) + \int_{-\infty}^{0} A\,(t,\theta)x(t+\theta)d\theta + \int_{0}^{\infty} A(t,\theta)\,x(t+\theta)d\theta$$

$$L^{*}(t,Z)x_t \;\; = \sum_{k=0}^{\infty} A_k\, x(t-w_k) + \int_{-\infty}^{0} A\,(t,\theta)x(t+\theta)d\theta.$$

The representation L, L^{*} *are the same under the following assumptions;*

$$L(t,Z)x_t \;\; = \underset{p\to\infty}{limit} \sum_{k=0}^{p} A_k\, x(t-w_k) + \underset{M.N\to\infty}{limit} \int_{M}^{N} A\,(t,\theta)x(t+\theta)d\theta.$$

We assume the limits exist, giving finite partial sum for the infinite series and the improper integrals. Thus the system

$$L^{*}(t,Z)x_t \;\; = \sum_{k=o}^{\infty} A_k\, x(t-w_k) + \int_{-\infty}^{0} A\,(t,\theta)x(t+\theta)d\theta$$

is finite and well defined function.

In the light of the above, the system (2.1) *reduces to*

$$\frac{d}{dt}[D(t,Z)x_t] = \;L(t,z)x_t \;+ \int_{-h}^{0} [d_\theta\, H(t,\theta)u(t+\theta) + \;f(t,x_t) \;\ldots\ldots\ldots(2.3)$$

$x(t_0) = \phi \in C.$

$$Where\;,\;\; L(t,Z)x_t \;\; = \sum_{k=0}^{p} A_k\, x(t-w_k) + \int_{-h}^{0} A\,(t,\theta)x(t+\theta)d\theta.$$

2.3 Variation of Constant Formula

Integrating system (2.3), *after linearizing, we have*

$$x(t) = X(t,t_0,\phi,u)x_0 + \int_{t_0}^{t} X\ (t,s)f(s,x_s)ds + \int_{0}^{t} X\ (t,s)\left\{\int_{-h}^{0} d_\theta\, H(t,\theta)u(t+\theta)\right\} ds \qquad (2.4)$$

Where $X(t,s)$ is the fundamental matrix of the homogeneous part of the system (2.3). $X(t,s) = 1$ *(identity matrix of nxn) order, for* $s = t$.

The 3rd term in the right hand side of system (2.4) contains the values of the control $u(t)$ for $t < t_0$, as well as for $t > t_0$. The values of the control $u(t)$ for $t \in [t_0 - h, t_0]$ enter into the definition of the initial complete state $z(0) = \{x_0, u_{t_0}\}$.

*To separate them, the 3rd term of system (2.4) must be transformed by changing the order of integration. Using the unsymmetric Fubini theorem (**see J. Klanka (1980)**), we have the following equalities:*

$$x(t) = X(t,t_0,\phi,u)x_0 + \int_{0}^{t} X\ (t,s)\, f(s,x_s)ds + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_o}^{t} X\,(t,s)H(s,\theta)u(s+\theta)ds\right) \ldots\ldots\ldots (2.5)$$

$$= X(t,t_0,\phi,u)x_0 + \int_{t_0}^{t} X\ (t,s)\, f(s,x_s)ds + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t+\theta} X\,(t,s-\theta)H(s-\theta,\theta)u(s-\theta+\theta)ds\right) \ldots\ldots\ldots.(2.6)$$

$$= X(t,t_0,\phi,u)x_0 + \int_{t_0}^{t} X\ (t,s)\, f(s,x_s)ds + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t+\theta} X\,(t,s-\theta)H(s-\theta,\theta)u(s)ds\right) \ldots\ldots (2.7)$$

$$= X(t,t_0,\phi,u)x_0 + \int_{t_0}^{t} X\ (t,s)\, f(s,x_s)ds \ + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} X\,(t,s-\theta)H(s-\theta,\theta)u_0(s)ds\right) + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0+\theta} X\,(t,s-\theta)H(s-\theta,\theta)u(s)ds\right) \ldots\ldots\ldots.(2.8)$$

Where the symbol d_{H_θ} denotes that the integration is in the Lebesque − Stieltjes sense with respect to the variable θ in the function $H(t,\theta)$.

Let us introduce the following notations:

$$\widehat{H}(s,\theta) = \begin{cases} H(s,\theta), for\ s \le t, \theta \epsilon R \\ 0, \quad for\ s > t, \theta \epsilon R \end{cases} \ldots\ldots..(2.9)$$

$Hence\ x(t)\ can\ be\ expressed\ in\ the\ following\ form:$

$$x(t) = X(t,t_0,\phi,u)x_0 + \int_{t_0}^{t} X\ (t,s)\ f(s,x_s)ds$$
$$+ \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} X\,(t,s-\theta)H(s-\theta,\theta)u_0(s)ds\right)$$
$$+ \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0}^{t} X\,(t,s-\theta)\widehat{H}(s-\theta,\theta)u(s)ds\right) \dots\dots (2.10)$$

$Using\ again\ the\ unsymmetric\ Fubini\ theorem, the\ equality\ (2.10) can\ be\ rewritten\ in\ a\ more\ convenient\ form\ as\ follows:$

$$x(t) = X(t,t_0,\phi,u)x_0 + \int_{t_0}^{t} X\ (t,s)\ f(s,x_s)ds$$

$$+ \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} X\,(t,s-\theta)H(s-\theta,\theta)u_0(s)ds\right)$$

$$+ \int_{t_0}^{t}\left[\int_{-h}^{0} X\,(t,s-\theta)d_{H_\theta}H(s-\theta,\theta)\right]u(s)ds \quad \dots\dots\dots (2.11)$$

$Now\ let\ us\ consider\ the\ solution\ x(t)\ of\ system\ (2.1)\ for\ t = t_1, to\ getting:$

$$x(t_1) = X(t,t_0,\phi,u)x_0 + \int_{t_0}^{t_1} X\ (t,s)\ f(s,x_s)ds$$

$$+ \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} X\,(t,s-\theta)H(s-\theta,\theta)u_0(s)ds\right)$$
$$+ \int_{t_o}^{t_1}\left[\int_{-h}^{0} X\,(t,s-\theta)d_{H_\theta}\widehat{H}(s-\theta,\theta)\right]u(s)ds \qquad \dots\dots..(2.12)$$

Now, let

$$\alpha(t) = X(t,t_0,\phi,u)x_0 + \int_{t_0}^{t_1} X\ (t,s)\ f(s,x_s)ds \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.(2.13)$$
$$\beta(t) = \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} X\,(t,s-\theta)H(s-\theta,\theta)u_0(s)ds\right) \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..(2.14)$$

and

$$\mu(t) = \int_{t_o}^{t_1}\left[\int_{-h}^{0} X\,(t,s-\theta)d_{H_\theta}\widehat{H}(s-\theta,\theta)\right]u(s)ds. \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.(2.15)$$

$Substituting\ (2.13), (2.14)\ and\ (2.15)\ into\ (2.12)\ we\ have$

$$x(t_1) = \alpha(t_1) + \beta(t_1)\ + \int_{t_0}^{t_1} \mu(t_1)u(s)ds \ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2.16)$$

$$= \mu(t_1) + \beta(t) + R(t_1,t_0).$$

Definition 2.1 (Reachable set)
The reachable set of the system (2.1) is given as

$$R(t_{1,}t_0) = \left\{\int_{t_o}^{t_1}\left[\int_{-h}^{0} X\,(t,s-\theta)d_{H_\theta}\widehat{H}(s-\theta,\theta)\right]u(s)ds : u\epsilon U; |u_j| \le 1\right\}$$

$Where, U = \{u \in L_2\,([0,t_1],E^m) and\ j = 1,2,3,\dots,m$

Definition 2.3 (Attainable Set)
The attainable set for the system (2.1)is the set of all possible solutions of the system (2.1). visa vis system (1.1). It is denoted by

$$A(t_{1,}t_0) = \{x(t_,x_0,u): u \in U\}, \qquad where\ U = \{u \in L_2([0,t_1],E^m): |u_j| \le 1\,; \mathrm{j} = 1,2,\dots,\mathrm{m}$$

Definition 2.4 (Target Set)

The target set of system (2.1) is given as

$$G(t_{1,}t_0) = \{x(t_{1,}x_0,u): t_1 \ge \tau > t_0\ for\ fixed\ \tau\ and\ u \in U\}$$
$$where, U = \{u \in L_2([0,t_{1,}],E^m): |u_j| \le 1, \quad j = 1,2,\dots\dots,m\}$$

Definition 2.5 (Controllability grammian)
The controllability grammian of the system (2.1) is given as

$$W(t_{1,}t_0) = \int_{t_0}^{t_1}\left[\left[\int_{-h}^{0} X(t_1,s-\theta)d_{H_\theta})\widehat{H}(s-\theta,\theta)\right]\right]\left[\left[\int_{-h}^{0} X(t_1,s-\theta)d_{H_\theta})\widehat{H}(s-\theta,\theta)\right]^T ds$$

Where T denotes matrix transpose.

Definition 2.6 (Complete state)
The complete state of system (2.1)is given as

$$Z(t) = \{x(t),u_t\}, while\ the\ initial\ comeplete\ state\ \ z(t_0) = \{x_0,u_0\}.$$

Definition 2.7 (Properness)
The system (2.1) is proper in E^n on $[t_{0,}t_1]$ if span $(t_{1,}t_0) = E^n$. That is if

$$C^T\left[\int_{-h}^{0} X\,(t,s-\theta)d_{H_\theta}\widehat{H}(s-\theta,\theta)\right] = 0\ \ a.e. \Longrightarrow C = 0, C \in E^n, t_1 > 0.$$

Definition 2.8 (Relative Controllability)
The system (Relative controllable on $[t_{0,}t_{1,}]$ if

$$A(t_{1,}t_0) \cap\ G(t_{1,}t_0) \ne \phi, t_1 > 0.$$

3. *Main Results*

Optimality conditions for the Nonlinear Infinitely Space of Neutral Differential systems with Distributed Delays in the Control. In these systems , the optimal control problem can best be understood in the context of a game of pursuit,
(**see K. Balachandran and J. P. Dauer and Ananhdi (2003)**). *The emphasis here is the*

search for a control energy that can steer the state of the system of our interest to the target set (which can be a moving point function or a compact set function) in a minimum time. In other words, the optimal control problem can be stated as follows:

$$If\ t^* = infinum\,\{t: A(t_{1,}t_0) \cap G(t_{1,}t_0) \neq \phi\ for\ t \in [t_{0,}t_1],\ t > 0\}.$$

Then there exists an admissible control u^* such that the solution of the system with this admissible control is steered to the target. The proposition that follows illustrates this assertion.

***Proposition* 3.1**
Consider the system (2.1)

$$\frac{d}{dt}\big(D(t,x_t)\big) = L(t,z)x_t + \int_0^\infty A(t,\theta)x(t+\theta)d\theta + \int_{-h}^0 d\theta H(t,\theta)u(t+\theta) + f(t,x_t) \quad \ldots\ldots\ldots\ldots..(2.1)$$

as a differential game of pursuit, with its basic assumptions.
Suppose that $A(t_{1,}t_0)$ and $G(t_{1,}t_0)$ are compact set functions, then there exists an admissible control u such that the state of the weapon for the pursuit of the target satisfies the system (2.1) *if and only if $A(t_{1,}t_0) \cap G(t_{1,}t_0) \neq \phi$.*

Proof
Let $\{u^n\}$be a sequence of controls in U. Since the constraint control U is compact, then the sequence $\{u^n\}$ has a limit u, as in tends to infinity. That is

$$\underset{n\to\infty}{limit}\ u^n = u \quad \ldots\ldots\ldots\ldots\ldots.(3.1)$$

Suppose that the state $y(t)$ of the weapon for the pursuit of the target satisfies the system (2.1) *on the time interval $[t_0\ ,t_1]$, then $y(t) \in G(t,0)$, for $t \in [t_0\,,t_1]$.*
We are to show that there exists solution $x(t,u) \in A\,(t_{1,}t_{0,})$, for $t \in [t_{0,}t_1]$ such that $y(t) = x(t,u)$ for some $u \in U$. Now, $x(t,u) \in A(t,0)$ and from

$$x(t_{1,}x_0,u^n) = X(t_{1,}t_{0,}\phi)x_0 + \int_{t_0}^{t_1} X\,(t,s)f(s,x_s)ds + \int_{-h}^0 d_{H_\theta}\Big(\int_{t_0+\theta}^{t_0} X\,(t_1,s-\theta)H(s-\theta,\theta)u_0^n(s)ds\Big) + \int_{t_o}^{t_1}\left[\int_{-h}^0 X\,(t_1,s-\theta)d_{H_\theta}\widehat{H}(s-\theta,\theta)\right]u^n(s)ds \quad \ldots\ldots\ldots(3.2)$$

Taking limit on both sides of the system (3.2), *We have*

$$\underset{n\to\infty}{limit}\, x(t_{1,}x_0,u^n) = \underset{n\to\infty}{limit}\, X\,(t_{1,}x_0,u^n)x_0$$

$$+\int_{t_0}^{t_1} X(t,s)f(s,x_s)ds + \int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} X(t_1,s-\theta)H(s-\theta,\theta)\lim_{n\to\infty}\text{it}\, u_0^n(s)ds\right)$$

$$+\int_{t_0}^{t_1}\left[\int_{-h}^{0} X(t_1,s-\theta)d_{H_\theta}\widehat{H}(s-\theta,\theta)\right]\lim_{n\to\infty}\text{it}\, u^n(s)ds,\ \ for\ t\in[t_0,t_1]$$

$$\Rightarrow\ x(t_{1,}x_0,u) = X(t_{1,}t_{0,}\phi,u)x_0 + \int_{t_0}^{t_1} X(t,s)f(s,x_s)ds$$

$$+\int_{-h}^{0} d_{H_\theta}\left(\int_{t_0+\theta}^{t_0} X(t_1,s-\theta)H(s-\theta,\theta)u_0(s)ds\right)$$

$$+\int_{t_0}^{t_1}\left[\int_{-h}^{0} X(t_1,s-\theta)d_{H_\theta}\widehat{H}(s-\theta,\theta)\right]u(s)ds$$

$$= x(t_{1,}x_0,u) \in A(t_{1,}t_0).$$

$$Since\ A(t_{1,}t_0)\ is\ compact\ and\ \lim_{n\to\infty}\text{it}\, x(t_{1,}x_0,u^n) = x(t_{1,}x_0,u), there\ exists\ a\ control$$

$$u\in U, such\ that\ x(t_{1,}x_0,u) = y(t_1), for\ t>0.$$

$$Since\ y(t_1)\in G(t_{1,}t_0)\ and\ also\ is\ in\ A(t_{1,}t_0), it\ follows\ that$$

$$A(t_{1,}t_0)\cap G(t_{1,}t_0)\neq\phi, for\ t\in[t_{0,}t_1]$$

Conversely,

Suppose that the intersection condition holds; $A(t,0)\cap G(t,0)\neq\phi, t\in[t_0,t_1]$, *then there exists* $y(t)\in A(t_1,t_0)$ *such that* $y(t)\in G(t_1,t_0), t_1>t_0$
This implies that $y(t) = x(t_{1,}x_0,u)$ *and hence establishes that the state of the weapon of pursuit of the target satisfies the system* (2.1). *This completes the proof.*

***Remark* 3.1**

The above stated and proved proposition in other words states that in any game of pursuit described by a nonlinear infinitely space of Neutral functional Differential Systems with distributed delays in control, it is always possible to obtain the control energy function to steer the system state to the target in finite time. The next theorem is, therefore, a consequence of this understanding and provides sufficient conditions for the existence of the control that is capable of steering the state of the system (2.1) *visa vis system* (1.1) *to the target set in minimum time.*

***Theorem* 3.1 (*sufficient conditions for the existence of optimal control*).**
Suppose that the system (2.1), *that is*

$$\frac{d}{dt}\big(D(t,x_t)\big) = L(L(t,z)x_t + \int_0^\infty A(t,\theta)x(t+\theta)d\theta$$

$$+\int_{-h}^{0} d\theta H(t,\theta)u(t+\theta) + f(t,x_t) \quad \ldots\ldots\ldots..(2.1)$$

$x(t_0) = \phi,$

is relatively controllable on the finite interval $t_0, t_1]$, *then there exists an optimal control.*

Proof

By the controllability of the system (2.1), *the intersection condition holds. That is,*

$$A\,(t_1,t_0) \cap G(t_1,t_0) \neq \phi.$$

Also as $x(t_1, x_0, u) \in G(t_1,t_0)$, *so* $y(t_1) = x(t_1, x_0, u)$.

Recall that the attainable set $A\,(t_{1,}t_{o,})$ *is a translation of the reachable set* $R(t_{1,}t_{o,})$ *through the origin* ŋ *which is given as*:

$$ŋ = X(t_{1,}t_{0,}\phi, u)x_0 + \int_{t_0}^{t_1} X\,(t,s)f(s,x_s)ds$$

$$+\int_{-h}^{0} d_{H_\theta}\Big(\int_{t_0+\theta}^{t_0} X\,(t_1, s-\theta)H(s-\theta,\theta)u_0(s)ds.$$

It follows that $y(t) \in A\,(t_1,t_0)$ *for* $t \in [t_0,t_1]$, $t > 0$ *and can be defined as*

$$y(t) = \int_{t_0}^{t}\left[\left[\int_{-h}^{0} X\,(t_1, s-\theta)d_{H_\theta}\widehat{H}(s-\theta,\theta)\right]u(s)ds\right. .$$

Let $t^* = infimum\,\{t: y(t) \in R(t_1, t_0), t \in [t_0, t_1]\}$

Now, $t > 0$ *and there is a sequence of times* $\{t_n\}$ *and the corresponding sequence of controls* $\{t_n\} \in U$ *with* $\{t_n\}$ *converging to* t^* (*the minimum time*).

Let $y(t_n) = z(t_n) = z(t_n, u^n)$

Let $y(t_n) = z(t_n, u^n) \epsilon R(t_1, t_0)$.

Also

$$|y(t^*) - z(t^*,u^n)| = |y(t^*) - y(t_n) + y(t_n) - z(t^*,u^n)|$$

$$\leq |y(t^*) - y(t_n)| + |y(t_n) - z(t^*,u^n)|$$

$$\leq |y(t^*) - y(t_n)| + |\,z(t_n,u^n) - z(t^*,u^n)|$$

$$\leq |y(t^*) - y(t_n)| + \int_{t^*}^{t_n} \|z(s)\|ds$$

By the continuity of $y(t)$ *which follows the continuity of* $R(t_1, t_0)$ *as a continuous set function and the integrability of* $\|z(t)\|$, *it follows that* $z(t^*,u^n) \to y(t^*)$ *as* $n \to \infty$,

where $y(t^*) = z(t^*,u^*) \in R(t_1,t_0)$.

For some $u^* \in U$ *and by definition of* t^*, u^* *is an optimal control.*

This establishes the existence of an optimal control for the nonlinear infinitely space of Neutral Functional Differential System with distributed delays in the control (*system* 2.1) /*system*(1.1).

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