LEAST-SQUARE PARABOLA APPROXIMATION OF OPTIMAL TIME REPLACEMENT OF INDUSTRIAL EQUIPMENT FOR DISCRETE MODEL

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ABSTRACT
The purpose of this paper is to develop an approximate method for determining the approximate optimal time to replace equipment whose repair and maintenance cost increases and their efficiency reduces with time for a discrete model. Furthermore, we established existence and uniqueness theorem via the least-square parabola method for the optimal replacement time obtained.

Keywords: Scrap Value, Average Annual Cost, Running Cost, Replacement Cost, Approximation and Discrete Model.

1.0 INTRODUCTION

Replacement Problem (RP) is one of the key decision areas in engineering economic policy. In our daily life, replacement of an equipment or item is needed to maintain our desired efficiency level. Once equipment is designed, fabricated and installed, the operational availability of the same is looked after by the maintenance requirement. The idea of maintenance is very old and is believed to have been introduced along with inception of the machine. In the early days, a machine was used as long as it worked. When it stopped working, it was either repaired or replaced. The replacement problems are concerned with the issues that arise when the performance of an item decreases, fail or breakdown occurs (Pranab and Surapati 2011, Ajibola et al 2014, Allen and Wai-Ki 2005).

The solution to replacement problem is nothing but arriving at the best policy that determines the time at which the replacement is most economical instead of continuing at an increased maintenance cost.

Modeling of the replacement problem has attracted the interest of many authors. Nakagawa and Osaki (1997) introduced the age replacement problem. Bellman (1955) studied RP with dynamic programming. Hartman and Rogers (2006) extended dynamic programming approaches for equipment replacement problems with continuous and discontinuous technological change. Thi et al (2010) discussed an optimal maintenance and replacement decisions under technological change. They used stochastic dynamic programming in dealing with the optimal maintenance and replacement policy of equipment as a function of performance and cost. Abdelwali et al (1997) discussed parametric multi-objective dynamic programming with application to automotive problems. Abdelwali et al (2011) studied optimum replacement policies and applied the concept for Kuwait Passenger Transport Company. They used dynamic programming technique for generating the optimal replacement policies for buses. Ahmed (2010) studied an algorithm for a deterministic RP. RP with different criteria were studied (Alchian 1952, Dreyfus and Law 1977, Wegner 1975, Oakford et al1984, Olnishi 1997, Dimitrakos and Kyriakidis 2007, and Mahdavi 2009).

The purpose of this paper is to develop an approximate method of determining the appropriate optimal time to replace the equipment whose repair and maintenance cost increase and their efficiency reduce with time for a discrete model. Furthermore, we establish some existence and uniqueness theorem governing this optimal replacement time.

2.0 PRELIMINARIES

In replacement policy of discrete time model for items whose running cost increases with time and value of money remains constant during a period, we denote

R_t: Running cost of equipment at time t.

C: Capital cost of equipment

S: Scrap (salvage) value of equipment at the end of year t.

Thus, the running cost incurred during the t_n years is

$$R_{t_n} = \sum_{t=t_1}^{t_n} R_t \qquad (1)$$

The total cost TC_t incurred on the equipment during the t_n years is

$$TC_{t_n} = \mathrm{C} - \mathrm{S} + \sum_{t=t_1}^{t_n} R_t \qquad (2)$$

and the average cost AC_t for the t_n years is given by

$$AC_{t_n} = \frac{1}{t_n}\left(\mathrm{C} - \mathrm{S} + \sum_{t=t_1}^{t_n} R_t\right) \qquad (3)$$

We seek for t_{n^*} that minimizes AC_{t_n}, for such t_{n^*} we must have

$$AC_{t_{n*}} < AC_{(t_{n^*}\pm 1)} \qquad (4)$$

For simplicity, we taken $t_n = n$ so that inequality (2.4) becomes

$$AC_{n^*} < AC_{(n^*\pm 1)} \qquad (5)$$

Now

$$AC_{(n^*+1)} = \frac{1}{n^*+1}\left[\sum_{t=1}^{n^*+1} R_t + M\right] : \quad \mathrm{M} = \mathrm{C} - \mathrm{S}$$

$$= \frac{1}{n^*+1}\left[\sum_{t=1}^{n^*} R_t + M + R_{(n^*+1)}\right]$$

$$= \frac{n^*}{n^*+1}\left[\frac{1}{n^*}\left(\sum_{t=1}^{n^*} R_t + M\right) + \frac{R_{(n^*+1)}}{n^*}\right]$$

$$= \frac{n^*}{n^*+1} AC_{n^*} + \frac{R_{(n^*+1)}}{n^*+1}$$

$$\Rightarrow \quad AC_{(n^*+1)} - AC_{n^*} = \frac{R_{(n^*+1)}}{n^*+1} - \frac{AC_{n^*}}{n^*+1}$$

$$\Rightarrow \quad R_{(n^*+1)} = AC_{n^*} = (n^* + 1)\left(AC_{(n^*+1)} - AC_{n^*}\right) > 0$$

$$\Rightarrow \quad R_{(n^*+1)} > AC_{n^*} \qquad (6)$$

Next, observe that

$$AC_{(n^*-1)} = \frac{1}{n^*-1}\left[\sum_{t=1}^{n^*-1} R_t + M\right]$$

$$= \frac{1}{n^* - 1}\left[\sum_{t=1}^{n^*} R_t + M - R_{n^*}\right]$$

$$= \frac{n^*}{n^* - 1}\left[\frac{1}{n^*}\left(\sum_{t=1}^{n^*} R_t + M\right) - \frac{R_{n^*}}{n^*}\right]$$

$$= \frac{n^*}{n - 1} AC_{n^*} - \frac{R_{n^*}}{n^* - 1}$$

$$= AC_{(n^*-1)} - AC_{n^*} = \frac{AC_{n^*}}{n^* - 1} - \frac{R_{n^*}}{n^* - 1}$$

$$AC_{n^*} - R_{n^*} = (n^* - 1)\,(AC_{(n^*-1)} - AC_{n^*}) > 0$$

$$\Rightarrow \quad R_{n^*} < AC_{n^*} \quad (7)$$

Hence, combining (2.6) and (2.7) we have

$$R_{n^*} < AC_{n^*} < R_{(n^*+1)} \quad (8)$$

This implies that the replacement of industrial equipment (machine) when the maintenance cost in the (n+1)th year becomes greater than the average annual cost in the nth year will minimize the average annual cost.

In applying inequality (8) to solve real life problem, the conventional approach is to construct a cumulative cost table over the period of years, and then obtain the average annual cost column from which we check for the cell that satisfy inequality (8). The time t in the cell of average annual cost column where inequality (8) is satisfied is chosen to be the optimal time replacement for the equipment. Beyond solving for such an optimal time using tables, in this paper we develop a robust method of determining the approximate optimal time replacement for industrial equipments and the established existence and uniqueness theorem of this optimal time replacement.

3.0 MAIN RESULTS/LEAST SQUARE PARABOLA APPROXIMATION METHOD

Lemma 3.1 (Existence Theorem) (David and Ward, 1990)

Let $f: [a, b] \longrightarrow R$ be a continuous function and $y \in (f(a), f(b))$ then there exists $x \in (a, b)$ such that $f(x) = y$.

Lemma 3.2 (Uniqueness Theorem) (David and Ward, 1990)

If f belongs to $C^2(R)$ is increasing, is convex and has a zero, then the Newton's iteration scheme converges uniquely to the zero of f.

Remark 3.3 A consequence of lemma 3.1as applied in this work, is the case where $y = 0$, as such f must have a zero in the interval (a, b). Since $f(a)f(b) < 0$, the function changes sign on the interval $[a, b]$, hence, there exists at least one zero $x \in (a, b)$ such that $f(x) = 0$.

Let $D = \{(t_j, R_j): j = 1, 2 \ \ldots, n\}$ be the set of points constituting the running cost R_j at the $t_j th$ year. We approximate the points in D by the equation

$$R_t = a_0 + a_1 j + a_2 j^2 : \quad (t_j = j) \quad (9)$$

$$= \sum_{k=0}^{2} a_k j^k$$

Where the constant a_0, a_1, and a_2 are determine by solving the system of equations.

$$\sum_{j=1}^{n} R_j = na_0 + a_1\sum_{j=1}^{n} j + a_2\sum_{j=1}^{n} j^2$$
$$\sum_{j=1}^{n} jR_j = a_0\sum_{j=1}^{n} j + a_1\sum_{j=1}^{n} j^2 + a_2\sum_{j=1}^{n} j^3$$
$$\sum_{j=1}^{n} j^2R_j = a_0\sum_{j=1}^{n} j^2 + a_1\sum_{j=1}^{n} j^3 + a_2\sum_{j=1}^{n} j^4$$

Let $S_r(n) = \sum_{j=1}^{n} j^r$ and $q_r(n) = \sum_{j=1}^{n} j^{r-1}R_j$

then the system of equation becomes

$$q_1 = a_0n + a_1S_1 + a_2S_2$$
$$q_2 = a_0S_1 + a_1S_2 + a_2S_3$$
$$q_3 = a_0S_2 + a_1S_3 + a_2S_4$$
$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} n & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

Then the matrix equation is

$$Q = SA \qquad (10)$$

the determinant of the matrix **S** denoted by Δ is given by

$$\Delta = |S| = \left(\prod_{j=1}^{3} S_j + \prod_{j=1}^{2} S_{2j}\right) + (S_2^3 + nS_3^2 + S_1^2S_4) \qquad (11)$$

and

$$\Delta a_0 = \begin{vmatrix} a_1 & s_1 & s_2 \\ a_2 & s_2 & s_3 \\ a_3 & s_3 & s_4 \end{vmatrix} = (q_1S_2S_4 + q_2S_2S_3 + q_3S_1S_3) - (q_1S_3^2 + q_2S_1S_4 + q_3S_2^2) \qquad (12)$$

$$\Delta a_1 = \begin{vmatrix} n & q_1 & s_2 \\ s_1 & q_2 & s_3 \\ s_2 & q_3 & s_4 \end{vmatrix} = (q_1S_1S_3 + q_2nS_4 + q_3S_1S_2) - (q_1S_1S_4 + q_2S_2^2 + q_3nS_3) \qquad (13)$$

$$\Delta a_2 = \begin{vmatrix} n & s_1 & q_1 \\ s_1 & s_2 & q_2 \\ s_2 & s_3 & q_3 \end{vmatrix} = (q_1S_1S_3 + q_2S_1S_2 + q_3nS_2) - (q_1S_2^2 + q_2nS_3 + q_3S_1^2) \qquad (14)$$

thus

$$a_k = \frac{\Delta a_k}{\Delta}: \quad (k = 0, 1, 2) \qquad (15)$$

where

$s_1(n) = \frac{1}{2}n(n+1)$; $S_2(n) = \frac{1}{6}n(n+1)(2n+1)$; $S_3(n) = \frac{1}{4}n^2(n+1)^2$;

$S_4((n) = \frac{1}{30}n(n+1)(2n+1)(3n(n+1) - 1)$

Now, substituting (2.9) into (2.1) yield

$$N_n = \sum_{t=i}^{n}\sum_{k=0}^{2} a_k t^k = \sum_{k=0}^{2} a_k S_k(n)$$
$$= n\left[a_0 + \frac{(n+1)}{2}\left(a_1 + \frac{(2n+1)}{3}a_2\right)\right]$$
$$= \left(a_0 + \frac{a_1}{2} + \frac{a_2}{6}\right)n + \left(\frac{a_1}{2} + \frac{a_2}{2}\right)n^2 + \frac{a_2}{3}n^3$$
$$= \beta_0 n + \beta_1 n^2 + \beta_2 n^3: \beta_0 = a_o + \frac{a_1}{2} + \frac{a_2}{6}, \beta_1 = \frac{a_1}{2} + \frac{a_2}{2}, \beta_2 = \frac{a_2}{3}. \qquad (16)$$

Thus,

$$TC_n = N_n + M$$

$$\Rightarrow \quad AC_n = \frac{M}{n} + \beta_0 + \beta_1 n + \beta_2 n^2$$

$$\frac{dAC_n}{dn} = \frac{-M}{n^2} + \beta_1 + 2\beta_2 n \tag{17}$$

Now, by (17) we let

$$\mathrm{H(n)} = \beta_1 n^2 + 2\beta_2 n^3 - M \tag{18}$$

Using the method of least-square parabola we wish to approximate n^* satisfying

$$AC_{n^*} < AC_{(n^*\pm 1)} \tag{19}$$

$$\left.\frac{dAC_n}{dn}\right|_{n=n^*} = 0 \tag{20}$$

Theorem 3.4

Let H(n) be given by (18) then there exist $n^* \in (n_\delta^*, n_\varepsilon^*,)$ for some $\varepsilon > 0, \delta > 0$ such that $\mathrm{H}(n^*) = 0$.

Proof

For $\varepsilon > 0$ *and* $\delta > 0$, we can chose n_ε^*, n_δ^* such that $\mathrm{H}(n_\varepsilon^*)>0$ and $\mathrm{H}(n_\varepsilon^*)<0$, where n_ε^* = inf {n: H(n)>0} and n_δ^* = sup {n: H(n) < 0}

Thus, by continuity of the function H and lemma 3.1, there exist $n^* \in(n_\delta^*, n_\varepsilon^*,)$ such that $\mathrm{H}(n^*) = 0$.

For such n^*, we must have

$$AC_{n^*} < AC_{n_\varepsilon^*} \text{ and } AC_{n^*} < AC_{n_\delta^*}$$

$$\Longrightarrow \quad AC_{n^*} < AC_{(n^*+1)} \text{ and } AC_{n^*} < AC_{(n^*-1)}$$

$$\Longrightarrow \quad AC_{n^*} < AC_{(n^*\pm 1)}$$

Thus, we use equation (20) to obtain approximate value of n^*. If a closed form solution does not exist for n^*, we can approximate n^* using any of the method for finding the zero of a non-linear equation. In particular, using the Newton's approximation method we define.

$$n_{j+1}^* = n_j^* - \frac{H(n_j^*)}{H^1(n_j^*)} \quad \forall\, j \geq 0 \tag{21}$$

where $n_0^* \in [n_\delta^*, n_\varepsilon^*,]$ is any initial guess that is sufficiently close to n* and satisfies

$$\left|n_{j+1}^* - n^*\right| \leq C\left|n_j^* - n^*\right| \quad \forall\, j \geq 0 \tag{22}$$

Furthermore, observe that

1. $H(n) \in C^2(R)$
2. H'' (n) > 0 ∀ n > 0 (H is convex on (0, ∞))
3. H'(n) > 0 ∀ n > 0 (H is increasing on (0, ∞))

CONCLUSION

Hence, by lemma 3.2 we conclude that the Newton's iteration scheme in (21) converge uniquely to $n^* \in (n^*_\delta, n^*_\varepsilon,)$ which is the optimal (Least-square parabola approximation) replacement time for the discrete model.

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