UNIQUENESS OF AN OPTIMAL CONTROL OF NONLINEAR INFINITE SPACE OF NEUTRAL FUNCTIONAL DIFFERENTIAL SYSTEMS WITH DISTRIBUTED DELAYS IN THE CONTROL.

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Abstract

In this paper, Nonlinear Infinitely Space of Neutral Functional Differential Systems with Distributed Delays in the Control of the form:

$$\frac{d}{dt}[D(t,x_t)] = L(t,x_t)x_t + \int_{-\infty}^{0} A(t)x(t+\theta)$$
$$+\int_{-h}^{0}[d_\theta H(t,\theta)]u(t+\theta) + f(t,x_t)$$

was presented for study. To obtain the solution of the system, we first of all linearize the nonlinear system for the work to be less tedious using Chukwu (1992)-like arguments. After linearization, we integrated the linearized system and obtained the mild solution of the system from which we extracted the set functions (reachable set, target set, attainable set) upon which our study hinged. Use was made of the Unsymmetric Fubuni Theorem to establish the results. It was established that the optimal control is unique. The establishment of the uniqueness of optimal control provided a new approach for the proof of the existence of an optimal control.

Keywords: Uniqueness, Optimal Control, Admissible Control, Reachable Set, Attainable Set.

1.0 INTRODUCTION

A neutral functional differential equation is given as a differential equation depending on past and present values which involve derivatives with delays (see Banks and Kent. (1972)). This means that the derivatives of the functional difference operator $D(t,x)$ is expressed in terms of the past and present states or the unknown functions. Neutral equations have been found to have great importance in various applications in natural sciences, technology and electrodynamics (see Balachandran and Dauer (2002) and Balachandran and Dawer (1996)). In his paper "Linear Systems with Delayed Control", Artstein (1982), the use of neutral equations in the study of electrical networks containing lossless transmission lines is made evident. Chukwu (2001) modeled a neutral dynamics to represent the rate of growth of the world economy. The existence and uniqueness conditions for neutral functional differential equation have been presented (see K. Balachandran (1992), Anichini and Zecca (1986)). Research on neutral systems has extended to finding necessary and sufficient conditions for the asymptotic behavior of solutions of such equations. Balachandran and Leelamani (2006), studies stability of infinite neutral systems.

The investigation here is the controllability of nonlinear infinite Neutral Differential Systems with distributed Delays in Control of the form:

$$\frac{d}{dt}[D(t,x_t)] = L(t,x_t)x_t + \int_{-\infty}^{0} A(t)x(t+\theta)$$

$$+\int_{-h}^{0}[d_\theta H(t,\theta)]u(t+\theta) + f(t,x_t) \qquad (1.1)$$

$x_{to} = \phi \quad (a\ nonlinear\ infinite\ neutral\ system).$

Our principal objective in this paper is to obtain necessary and sufficient conditions for controllability, and existence of an optimal control for the system (1.1) above. It is known from Onwuatu, J.U.(1993) that, if a system is relatively controllable, then optimal control is unique and bang-bang. In the light of this, we shall consider the nonlinear infinitely space of Neutral Differential Systems with Distributed Delays in Control of the form:

$$\frac{d}{dt}[D(t,x_t)] = L(t,x_t)x_t + \int_{-\infty}^{0} A(t)x(t+\theta)$$

$$+\int_{-h}^{O}[d_\theta H(t,\theta)]u(t+\theta) + f(t,x_t) \qquad (1.1)$$

The above system (system (1.1)) will be investigated for existence and uniqueness of optimal control by first of all considering the relative controllability of the system.

1. **Notation and Preliminaries**

Let n be a positive integer and $E = (-\infty,\infty)$ *be the real line. Denote by* E^n *the space of real n − tuples called the Euclidean space with norm denoted by* $|.|$.

$x(t_0) = \phi = x_0$ *is initial condition /function, where* $x \in E^n$ *is the state space and* $u \in E^m$ *is the control function,* $(H(t,\theta)$ *is an nxm matrix continuous at t and of bounded variation in* θ *on* $[-h,0]; h > 0$ *for each* $t \in [t_{0,}t_1]$;

$t_1 > t_{0.}$ *The nxn matrix* $A(t)$ *is continuous in its argument if* $J = [t_{0,}t_1]$ *is any interval of* E, L_2 *is the Lebesque space of square integrable functions from J to* E^n *written in full as* $L_2([t_{0,}t_1], E^n)$.

Let $h > 0$ *be a positive real number and let* $c([-h,0],E^n)$ *be a Banach space of continuous functions with the norm of uniform convergence defined by* $\|\phi\| = \sup \phi(s);\ -h \le s \le 0, for\ \phi \in C([-h,0],E^n)$

If x is a function from $[-h,0]$ *to* E^n*, then* x_t *is a function defined on the delay interval* $[-h,0]$ *given as*

$$\mathrm{x}_t(s) = x(t+s); s \in [-h,0], t \in [0,\infty).$$

Consider the nonlinear infinite neutral system with distributed delays in control,

$$\frac{d}{dt}[D(t,x_t)] = L(t,x_t)x_t + \int_{0}^{\infty} A(t)\,x(t+\theta)d\theta$$

$$+\int_{-h}^{0}[d_\theta\, H(t,\theta)]u(t+\theta) \qquad (2.1)$$

Where, $\quad L(t,x_t) = \sum_{k=0}^{\infty} A_k\,(x - w_k) + \int_{-\infty}^{0} A\,(t)x(t+\theta)d\theta.$

$$L(t,x_t) = \int_{-\mathrm{h}}^{0} \mathrm{d}_\theta\, \eta\big(t,s,x(t+s)\big)x(t+\theta)$$

$$\eta(t,s,\phi,\Psi) \ge 0, for\ \ s \ge 0\ and\ \phi,\Psi\ \in C.$$

$$\eta(t,s,\phi,\Psi) = \eta(t,s,\phi,\Psi), for\ \ t < -h.$$

$\eta(t, s, \phi, \Psi)$ *is a continuous matrix function of bounded variation in* $s \in [-h, 0]$, *vary* $\eta(t) \leq m(t), m(t) \in L_1$.

where L_1 *is the space of integrable functions.*

Let Ω *be an open subset of ExC and D and L be bounded linear operators defined on ExC into* E^n.

$$D(t, x_t) = x(t)g(t, x_t),$$

$$where,\ g(t, x_t) = \sum_{n=0}^{\infty} A_n(t)\phi(t - w_n(t) + \int_{-h}^{0} A(t, s)\phi(s)ds$$

$$= \int_{-h}^{0} d_\theta\, H(t, \theta)\phi(\theta)$$

$$Where,\ 0 \leq w_n \leq h, and \left|\int_{-h}^{0} d_\theta\, H(t, \theta)\phi(\theta)\right| \leq h(\theta)\|\phi\|.$$

$$D(t, x_t)\ is\ non-atomic\ at\ zero$$

$$(\boldsymbol{i.e\ differentiable\ and\ integrable\ at\ zero}).$$

$$\int_{-h}^{0} A(t, s)ds + \sum_{n-1}^{\infty} |A_n(t)| \geq \delta(\varepsilon), for\ all\ t, Where\ ,\delta(\varepsilon) \to 0.$$

f is continuous and satisfies other smoothness conditions.

***Linearization of the System* (2.1)**

Consider system (2.1) below

$$\frac{d}{dt}[D(t, x_t)] = L(t, x_t)x_t + \int_{0}^{\infty} A(t)\, x(t + \theta)d\theta$$

$$+ \int_{-h}^{0} [d_\theta\, H(t, \theta)]u(t + \theta) + f(t, x_t) \qquad (2.1)$$

$$(\boldsymbol{Circularity\ of\ the\ function\ from\ -\infty\ to\ 0, and\ from\ 0\ to\ \infty})$$

We can linearize the system (2.1) *as in Chukwu* (1992)*by setting* $x_t = z$ *in L, a specified function inside the function* $L(t, x_t)x_t$ *to have* $L(t, z)x_t$ *without loss of generality. Thus the system* (2.1) *becomes*

$$\frac{d}{dt}[D(t, x_t)] = L(t, z)x_t + \int_{0}^{\infty} A(t)\, x(t + \theta)d\theta$$

$$+ \int_{-h}^{0} [d_\theta\, H(t, \theta)]u(t + \theta) + f(t, x_t) \qquad (2.2)$$

Evidently

$$L(t,z)x_t = \sum_{k=0}^{\infty} A_k\, x(t-w_k) + \int_{-\infty}^{0} A\,(t)x(t+\theta)d\theta$$
$$+ \int_{0}^{\infty} A(t)\, x(t+\theta)d\theta$$

$$L^{*}(t,z)x_t = \sum_{k=0}^{\infty} A_k\, x(t-w_k) + \int_{-\infty}^{0} A\,(t,\theta)x(t+\theta)d\theta$$

$The\ representation\ L, L^{*}\ are\ the\ same\ under\ the\ following\ assumptions.$

$$L(t,z)x_t = \lim_{p\to\infty} \sum_{k=0}^{\infty} A_k\, x(t-w_k) + \lim_{M.N\to\infty} \int_{M}^{N} A\,(t,\theta)x(t+\theta)d\theta$$

We assume the limits exist, giving finite partial sum for the infinite series and the improper integrals. Thus the system

$$L^{*}(t,z)x_t = \sum_{k=0}^{\infty} A_k\, x(t-w_k) + \int_{-\infty}^{0} A\,(t,\theta)x(t+\theta)d\theta$$

$is\ finite\ and\ well\ defined\ function.$

$In\ the\ light\ of\ the\ above, the\ system\ (2.1)\ reduces\ to$

$$\frac{d}{dt}[D(t,x_t)] = L(t,Z)x_t + \int_{-h}^{0} [d_\theta\, H(t,\theta)u(t+\theta) + f(t,x_t) \qquad (2.3)$$

$x(t_0) = \phi \in C.$

$$Where\ , L(t,z)x_t = \sum_{k=0}^{p} A_k\, x(t-w_k) + \int_{-h}^{0} A\,(t,\theta)x(t+\theta)d\theta$$

Variation of Constant Formula

Integrating system (2.3), after linearizing, we have

$$x(t) = x(t,t_0,\phi,u) + \int_{0}^{t} \mathrm{x}\,(t,s)\{\int_{-h}^{0} [d_\theta\, H(t,\theta)]u(t+\theta)\}ds$$
$$+ \int_{0}^{t} \mathrm{x}\,(t,s)f(s,x_s)ds \qquad (2.4)$$

$Where\ X(t,s) is\ the\ fundamental\ matrix\ of\ the\ homogeneous\ part\ of\ the$
$system\ (2.3).\ \ X(t,s) = 1\ (identity\ matrix);\ t = s.$

$The\ 2nd\ term\ in\ the\ right\ hand\ side\ of\ system\ (2.4) contains\ the\ values\ of$
$the\ control\ u(t)\ for\ t < t_0, as\ well\ as\ for\ t > t_0.$

$The\ values\ of\ the\ control\ u(t) for\ t \in [t_0 - h, t_0] enter\ into\ the\ definition$
$of\ the\ initial\ complete\ state\ u_{t_0}.\ To\ separate\ them, the\ 2nd\ term\ of\ system\ (2.4) must\ be$

transformed by changing the order of integration. Using the unsymmetric Fubini theorem, we have the following equalities:

$$x(t) = x(t, t_0, \phi, U) + \int_0^t \mathrm{x}(t, s)\, f(s, x_s) ds$$

$$+ \int_{-h}^{0} d_{H_\theta}[\int_{t_0+\theta}^{t-\theta} X(t, s-\theta) H(s-\theta, \theta) u(s+\theta-\theta) ds]$$

$\Longrightarrow$

$$x(t) = x(t, t_0, \phi, U) + \int_{t_0}^{t} \mathrm{x}(t, s)\, f(s, x_s) ds$$

$$+ \int_{-h}^{0} d_{H_\theta}[\int_{t_{+\theta}}^{t_0} X(t, s-\theta) H(s-\theta, \theta) u_0(s) ds]$$

$$+ \int_{-h}^{o} d_{H_\theta}[\int_{t_o}^{t} X(t, s-\theta) H(s-\theta, \theta) u(s) ds]$$

Where the symbol d_{H_θ} denotes that the integration is in the Lebesque-Stieltjes sense with respect to the variable θ in the function $H(t, \theta)$.

Let us introduce the following notations:

$$\widehat{H}(s, \theta) = \begin{Bmatrix} H(s, \theta), for\ s < t, \theta \epsilon R \\ 0, \quad for\ s < t, \theta \epsilon R \end{Bmatrix}$$

Hence $x(t)$ *can be expressed in the following form*:

$$x(t) = x(t, t_0, \phi, U) + \int_{t_0}^{t} \mathrm{x}(t, s)\, f(s, x_s) ds$$

$$+ \int_{-h}^{0} d_{H_\theta}[\int_{t_{+\theta}}^{t_0} X(t, s-\theta) H(s-\theta, \theta) u_0(s) ds]$$

$$+ \int_{-h}^{0} d_{H_\theta}[\int_{t_0}^{t} X(t, s-\theta) \widehat{H}(s-\theta, \theta) u(s) ds] \qquad (2.6)$$

Using again the unsymmetric Fubini theorem, the equality (2.6) *can be rewritten in a more convenient form as follows*:

$$x(t) = x(t, t_0, \phi, U) + \int_0^t \mathrm{x}(t, s)\, f(s, x_s)$$

$$+ \int_{-h}^{0} d_{H_\theta}[\int_{\theta}^{0} X(t, s-\theta) H(s-\theta, \theta) u_0(s) ds]$$

$$+ \int_0^t [\int_{-h}^{0} X(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta)] u(s) ds \quad , t_0 = 0.$$

$Now\ let\ us\ consider\ the\ solution\ x(t)\ of\ system\ (2.1)\ for\ t = t_1, we\ have$

$$x(t_1) = x(t, t_0, \phi, U) + \int_0^t x\,(t,s)\, f(s, x_s) ds$$

$$+ \int_{-h}^{0} d_{H_\theta} [\int_\theta^0 X\,(t_1, s-\theta) H(s-\theta, \theta) u_{t_0} ds]$$

$$+ \int_0^t [\int_{-h}^0 X\,(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta)] u(s) ds \qquad (2.7)$$

$For\ brevity,$

$$let\ \beta(t) = x(t, t_0, \emptyset, U) + \int_0^t x\ (t,s)\, f(s, x_s) ds,$$

$$\mu(t) = \int_{-h}^0 d_{H_\theta} [\int_\theta^0 X\,(t_1, s-\theta) H(s-\theta, \theta) u_0(s) ds],$$

and

$$Z(t) = \left[[\int_{-h}^0 X\,(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) \right] So\ that$$

$$x(t, t_0, U) = \beta(t) + \mu(t) + Z(t_1, s) u(s) ds. \qquad (2.8)$$

2.3 *Stability Definitions*

$We\ now\ define\ the\ following:$

Definition 2.1 (Stability)

$The\ trivial\ solution\ x = 0\ of\ system\ (2.1) is\ stable\ if\ for\ any\ given\ to\ \in E, and\ a\ positive\ number\ \varepsilon > 0, there\ exists\ \delta = \delta(t_0 \epsilon)\ such\ that\ \phi \in \beta(0, \epsilon)\ implies\ that$

$$x_t(t_0, \phi) \in \beta(0, \delta\ )$$

$for\ all\ t \geq t_0, \phi \in C\ and\ \beta(0, r)\ is\ an\ open\ ball\ centered\ at\ 0, with\ radius\ r.$

Definition 2.2 (Uniform stability)

$The\ trivial\ solution\ x = 0\ of\ system\ (2.1) is\ uniform\ stable\ if\ for\ any\ \varepsilon > 0,\ thereexists\ \delta = \delta(\varepsilon)\ (independent\ of\ t_0\ such\ that\ \phi \in \beta(0, \varepsilon)\ implies\ that$

$$x_t(t_0, \phi) \in \beta(0, \delta), for\ all\ t > t_0.$$

Definition 2.3 (Asymptotic stability)

$The\ trivial\ solution\ x = 0\ of\ system\ (2.1) is\ asymptotically\ stable,\ if\ it\ is\ stable\ such\ that\ \phi \in \beta(0, \delta), implies\ that\ x_t(t_0, \phi) \to 0, as\ t \to \infty.$

Definition 2.4 (Uniform Asymptotic stability)

$The\ trivial\ solution\ x = 0\ of\ system\ (2.1) is\ uniformly\ asymptotically\ stable\ if\ the\ system\ is\ uniformly\ stable\ and\ for\ \phi \in \beta(0, \delta), implies\ that\ x_t(t_0, \phi) \to 0\ as\ t \to \infty.$

$The\ solution\ x_t(t_0,\phi) of\ system\ (2.1) is\ exponentially\ asymptotically\ stable\ it\ there\ exist\ constants\ k>0\ and\ c>0\ such\ that\ the\ solution\ satisfies\ x_{t_0}(t_0,\phi)=0\ and$

$$|x_t(t_0,\phi)| \leq ke^{t-t_0}.$$

***Definition* 2.5 (*Complete state*)**

It is known that the complete state of system (2.1) *at time t given by*

$$z(t) = \{x(t), u_t\}.$$

Then the initial complete state of the system (2.1)*is given by*

$$z(t) = \{x_0, u_{t_0}\}.$$

2.4 *BASIC SET FUNCTIONS AND PROPERTIES*

We shall define the set functions upon which our study hinges.

***Definition* 2.6(*Reachable set*)**

The reachable set of the system (2.1) *is given as*

$$R(t_1, t_0) = \left\{ \int_{t_o}^{t_1} \left[\int_{-h}^{0} x(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) u(s) \right] ds : |u_j| \leq 1, \quad j = 1,2,3, \dots \dots m. \right\}$$

***Definition* 2.7 (*Attainable Set*)**

The attainable set for the system (2.1) *is given as*

$$A(t_1, t_0) = \{x(t, x_0, u) : u \in U\}, \qquad where\ U = \{u \in L_2([0, t_1], E^m) : |u_j \leq 1|, \qquad j = 1,2, \dots \dots m\}.$$

It is a set of all possible solutions of the system (2.1).

Definition 2.8 (Target Set)

The target set of system (2.1) denoted by $G(t_1, t_0)$ is given as

$G(t_1, t_0) = \{x(t, x_0, u) : t_1 \geq T > t_0$, for fixed T *and* $u \in U\}$ where

$$U = \{u \in L_2([t_0, t_1], E^m) : |u_j| \leq 1 ; j = 1,2, \dots \dots m\}$$

Definition 2.9 (Controllability grammian)

The controllability grammian is given as

$$W(t_1, t_0) = \int_{t_0}^{t_1} Z(t,s) Z^T(t,s) ds$$

$$= \int_{t_0}^{t_1} \left[\int_{-h}^{0} x(t_1, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) \right] \left[\int_{-h}^{0} x(t_1, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) \right]^T$$

Where T denotes matrix transpose, and, $\text{Z} = \int_{-h}^{0} \boldsymbol{x(t_1, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta)}.$

2.5: *Relationship Between the Set Functions*

We shall first establish the relationship between the attainable set and the reachable set to enable us see that once a property has been proved for one set, then it is applicable to the other.
From equation (2.8),

$A(t_1, t_0) = \{\eta(t) + R(t_1, t_0)\}$ *for* $u \in U, t \in [t_0, t_1]$, *Where* $\eta(t) = \beta(t) + \mu(t)$.

This means that the attainable set is the translation of the reachable set through $\eta \in E^n$. Using the attainable set, therefore, it is easy to show that the set functions possess the properties of convexity, closeness and compactness. Also, the set functions are continuous on $[0, \infty]$ to the metric space of compact subsets of E^n, (see E. N. Chukwu (1988), I. Gyori (1982).This gives impetus for adaptations of the proofs of these properties for system (2.1).
Applications will be made of the following controllability conditions (controllability standard) to establish results:

1 The controllability Grammian or map $W(t_1, t_0)$ is invertible and the invertibility of the grammian means that the rank of the grammian must be equal to n.

a. *i.e Rank* $W(t_1, t_0) = n \Rightarrow W^{-1}(t_1, t_0)$ *exists.*

2 The non-emptiness of the intersection of two set functions- attainable set $A(t_1, t_0)$ and target set $G(t_1, t_0)$ is equivalent to the controllability of the system of interest.
That is, $A(t_1, t_0) \cap G(t_1, t_0) \neq \phi$, implies that the system of interest is controllable.

3 Zero in the interior of the reachable set implies that system of interest is controllable. That is

$$0 \in Int\ A(t_1, t_0) \Rightarrow \text{controllability of the system.}$$

4 *The system* (2.1) *is proper in* E^n *on* $[t_0, t_1]$, *if span* $R(t_1, t_0) = E^n$.
Or the system (2.1) *is proper (controllable) if the controllable index*

$$C^T \text{Z}(t, s) = 0 \Rightarrow C = 0$$

That is, if $c^T \left[\int_{-h}^{0} x(t_1, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta)\right] = 0$ *a.e*, $\Rightarrow c = 0, c \in E^n, t_1 > 0$.

5 *System* (2.1) *is relatively controllable on* $[t_0, t_1]$ *if,*

$$A(t_1, t_0) \cap G(t_1, t_0) \neq \phi\ ; t_1 > t_0.$$

***Definition* 2.10 (*Controllability Grammian or the nxn controllability matrix of* (2.1))**

Let

$$W(t_1, t_0) = \int_{t_0}^{t_1} \left[\int_{-h}^{0} x(t_1, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta)\right] \left[\int_{-h}^{0} x(t_1, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta)\right]^T$$

Where the symbol T denotes the matrix transpose.

3. *Main Results*

Here, a new method of approach is derived for the proof of the existence of optimal control of our system of interest.

Theorem 3.1. (Necessary Condition)

The following are equivalent for system (2.1) vis-a-via system (1.1) to be controllable:

(1) $w(t_1, t_0)$ is non-singular for each $t\epsilon[t_0, t_1]$.

(2) The system (2.1) vis-a-via system(1.1) is proper in E^n for each interval $[t_0, t_1]$,

(3) The system (2.1) vis-a-via system(1.1) is relatively controllable on each interval $[s_0, s]$

Proof

Recall that $\omega(t_1 , t_0)$ is non-singular is equivalent to saying that $\omega(t_1 , t_0)$ is positive definite, which in turn is equivalent to

$$C^T \int_{t_0}^{t_1} \left[\int_{-h}^{0} X(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) \right] = 0$$

a.e on $[t_0, t_1]$, $implies\ c = 0$.

Therefore, showing that (1) and (2) are equivalent.

To show that (2) and (3) are equivalent:

$$C^T \int_{t_0}^{t_1} \left[\int_{-h}^{0} X(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) \right] = 0 \ \ a.e.\ t \in [t_0, t_1], for\ each\ \ t\ , then$$

$$C^T \int_{t_0}^{t_1} \left[\int_{-h}^{0} X(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) \right] u(t)dt = C^T \int_{t_0}^{t_1} \left[\int_{-h}^{0} X(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) \right] u(t)dt$$

$for\ u \in L_2$

It follows from this that C is orthogonal to the set

$$R(t_1, t_{0,}) = \left\{ \int_{t_0}^{t_1} \left[\int_{-h}^{0} X(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) \right] u(t)dt\ ; |u_j| \leq 1\ , j = 1,2,3, \ldots, m. \right\}$$

If we assume the relative controllability of system (2.1) now, then,$R(t_1, t_0) = E^n$, so that $C = 0$, showing that (3) implies (2).

Conversely, assume system (2.1)/ (3.1) **is not controllable** so that $R(t_1, t_0) \neq E^n\ , for\ t_1 > t_0$. $Then\ ,\ there\ exists\ C \neq 0, C \in E^n, such\ that$ $\boldsymbol{C^T\ R(t_1, t_0) = 0}$.

It now follows that for all admissible controls $u \in L_2$ that

$$0 = C^T \int_{t_0}^{t_1} \left[\int_{-h}^{0} X(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) \right] u(t)dt$$

$$= \int_{t_0}^{t_1} c^T \left[\int_{-h}^{0} x(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) \right] u(t) dt$$

Hence,

$$c^T \int_{t_0}^{t_1} \left[\int_{-h}^{0} x(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) \right] u(t) dt = 0 \quad a.e, s \in [t_0, t_1], C \neq 0.$$

This, by definition of properness implies the system is not proper, since $C^T \neq 0$. Hence the system is relatively controllable.

Theorem 3.2

$$\frac{d}{dt}[D(t,x)] = L(t, xt)x_t + \int_{-\infty}^{o} A(t,\theta)x(t+\theta)d\theta + \int_{-h}^{0}(d_\theta H(t,\theta)u(t+\theta) \qquad (3.1)$$

with its standing hypothesis.

Suppose that u^* is the optimal control, then it is unique.

Proof

Let u^* and v^* be optimal controls for the system (3.1) visa vis system (1.1), then u^* and v^* maximize.

$c^T \int_{t_o}^{t_1}[\int_{-h}^{0} X(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta)$, for $t \in [o, t_1]$; $t_1 > o$,

Over all admissible controls $u \in U$, and also we have the inequality with u^* as the optimal control.

$$c^T \int_0^t \left[\int_{-h}^{0} X(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) \right] u(s) ds$$

$$\leq \int_0^{t^*} [\int_{-h}^{0} X(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) u^*(s) ds \qquad (3.2)$$

Also, using v^* as optimal control, we have

$$c^T \int_0^t \left[\int_{-h}^{0} X(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) \right] u(s) ds$$

$$\leq \int_0^{t^*} \left[\int_{-h}^{0} X(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) \right] v^*(s) ds \quad (3.3)$$

Taking maximum of u, over $[-1, 1]$, the range of definition of u^* in system (3.2) and (3.3), we have

$$c^T \int_0^t \left[\int_{-h}^{0} \mathrm{X}\,(\mathrm{t},\mathrm{s}-\theta)\mathrm{d}_{\mathrm{H}_\theta}\widehat{H}(s-\theta,\theta)\right] max|u(s)|ds;\ 1 \le s \le 1,$$

$$= c^T \int_0^t \left[\int_{-h}^{0} \mathrm{X}\,(\mathrm{t},\mathrm{s}-\theta)\mathrm{d}_{\mathrm{H}_\theta}\widehat{H}(s-\theta,\theta)\right] u^*(s)ds, \qquad for\ u, u^* \in U. \qquad (3.4)$$

Also,

$$c^T \int_0^t \left[\int_{-h}^{0} \mathrm{X}\,(\mathrm{t},\mathrm{s}-\theta)\mathrm{d}_{\mathrm{H}_\theta}\widehat{H}(s-\theta,\theta)\right] max|u(s)|ds;\ 1 \le s \le 1$$

$$= c^T \int_0^t \left[\int_{-h}^{0} \mathrm{X}\,(\mathrm{t},\mathrm{s}-\theta)\mathrm{d}_{\mathrm{H}_\theta}\widehat{H}(s-\theta,\theta)\right] v^*(s)ds, \qquad for\ u, v^* \in U. \qquad (3.5)$$

$v^*(s)$ being optimal control such that $-1 \le s \le 1$.

$$\boldsymbol{Subtracting\ equation}\ (\mathbf{3.5})\ \boldsymbol{from\ equation}\ (\mathbf{3.4}), \boldsymbol{we\ have}$$

$$c^T \int_0^t \left[\int_{-h}^{0} \mathrm{X}\,(\mathrm{t},\mathrm{s}-\theta)\mathrm{d}_{\mathrm{H}_\theta}\widehat{H}(s-\theta,\theta)\right] max|u(s)|ds;\ 1 \le s \le 1$$

$$- c^T \int_0^t \left[\int_{-h}^{0} \mathrm{X}\,(\mathrm{t},\mathrm{s}-\theta)\mathrm{d}_{\mathrm{H}_\theta}\widehat{H}(s-\theta,\theta)\right] max|u(s)|ds;\ 1 \le s \le 1$$

$$= c^T \int_0^t \left[\int_{-h}^{0} \mathrm{X}\,(\mathrm{t},\mathrm{s}-\theta)\mathrm{d}_{\mathrm{H}_\theta}\widehat{H}(s-\theta,\theta)\right] u^*(s)ds, \qquad for\ u, v^* \in U.$$

$$- c^T \int_0^t \left[\int_{-h}^{0} \mathrm{X}\,(\mathrm{t},\mathrm{s}-\theta)\mathrm{d}_{\mathrm{H}_\theta}\widehat{H}(s-\theta,\theta)\right] v^*(s)ds, \qquad for\ u, v^* \in U.$$

$$\Rightarrow 0 = c^T \int_0^t \left[\int_{-h}^{0} \mathrm{X}\,(\mathrm{t},\mathrm{s}-\theta)\mathrm{d}_{\mathrm{H}_\theta}\widehat{H}(s-\theta,\theta)\right] \{u^*(s) - v^*(s)\}ds \Rightarrow 0$$

$$= [\int_{-h}^{0} \mathrm{X}\,(\mathrm{t},\mathrm{s}-\theta)\mathrm{d}_{\mathrm{H}_\theta}\widehat{H}(s-\theta,\theta)]\{u^*(s) - v^*(s)\}ds$$

$$\Rightarrow u^*(s) - v^*(s) = 0$$

$$\Rightarrow u^*(s) = v^*(s).$$

This establishes the uniqueness of the optimal control for the system (3.1) visa vis system (1.1).

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