ON c(n)-GEOMETRIC DISTRIBUTION AND SOME OF ITS PROPERTIES

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ABSTRACT
This work presents a new distribution function that modifies and generalize the classical geometric distribution which identified as the $c(n)$-geometric distribution (simply n-geometric distribution if $n \geq 0$ and reverse n-geometric distribution if $n < 0$.) for every $n \in Z$. The associated cumulative distribution, generating functions and statistics were also established. The results obtained improve, generalizes and complement the works of several authors in the literature.

Keywords: Geometric Distribution, Generating Functions, Discrete Random Variables.

1.0 INTRODUCTION AND PRELIMINERIES

Philippou, Georghiou and Philippou [1] introduced the distribution of the number of trials until the first occurrence of consecutive k successes in Bernoulli trials with success probability p which they called a geometric distribution of order k. This distribution have caught the interested of researchers, owning to the work of Philippou et al. [1] whose contribution in this field seems to be very important. Philippou et al. [1] studied a geometric distribution of order k and defined a Poisson distribution of order k and a negative binomial distribution of order k, attention has been paid to interrelationships among the so called discrete distributions of order k and their exact distribution theory has been developed extensively by many researchers.

Ever since then, related concepts has been introduced, reliability theory of the consecutive-k-out-of-n:F systems and studied by Kontoleon [2], Chiang and Niu [3] and Derman, Lieberman and Ross [4]. The relationships between the reliability of the system and discrete distributions of order k were studied by Lambiris and Papastavridis [5], Fu [6,7,8], Fu and Beihua [9], Aki [10], Hirano [11], Philippou [12], Papastavridis [13], Aki and Hirano [14,15,16,17], Chrysaphinou and Papastavridis [18], Fu and Koutras [19,20], Godbole [21,22], Griffith [23], Hirano and Aki [24], Charalambides [25] and Makri and Philippou [26]. Various modifications have been made also on the underlying sequence, e.g. replacing the Bernoulli trials for other random sequences such as some urn models, a binary sequence of order k as in Aki [10], Aki and Hirano [27,28], Dhar and Jiang [29] and Balakrishnan [30].

It is quite interesting how the classical geometric distribution and related discrete distributions has been generalized and developed by above authors. However generalization of classical geometric distribution is by no mean exhaustive, hence, in this research work, we toe a different direction and introduce a new distribution function that modifies and generalize the classical geometric distribution which we shall call the c(n)-geometric distribution function for every n ∈ Z, which is somewhat due to some weighted or scale function. This is actually the case of most distribution functions in literature that generalizes the classical geometric distribution.

Random variable X is said to have a geometric distribution if the probability mass function is given by

$$\mathrm{f(x)} = (1-\mathrm{q})\mathrm{q}^{\mathrm{x}-1};\ \mathrm{x} = 1,2,\dots;\ 0 \leq \mathrm{q} \leq 1 \qquad (1.1)$$

or

$$f(x) = (1-q)q^{x};\ \ x = 0,1,\ldots;\ 0 \leq q \leq 1 \qquad (1.2)$$

with the corresponding cumulative distribution

$$F(x) = 1 - q^{x} \qquad (1.3)$$

and

$$F(x) = 1 - q^{x+1} \qquad (1.4)$$

Geometric distribution is one of the discrete probability functions that one often comes across in most standard text and journal papers.
Let $\varphi: D \to D \subset \mathbb{R}$ be a real-value function on $D = [0,1]$. observe that (1.1) and (1.2) could take the general form

$$f(x;q) = \varphi(q)q^{x-1};\ \ x \geq c \qquad (1.5)$$

To see this, if we take $\varphi(q) = 1 - q$ or $\varphi(q) = (1-q)q$, then equation (1.5) reduces to (1.1) or (1.2) which are special cases.
Furthermore, let

$$\psi: (\mathbb{N} \cup \{0\}) x\, D \to D$$

Where $\mathbb{N}$ is the set of natural numbers. Then, for all $n \in \mathbb{N} \cup \{0\}$ we define

$$f(x;n,q) = \psi(n,q)q^{x-1};\ \forall\, x \geq c(n). \qquad (1.6)$$

Equation (1.6) is more general than equation (1.5), observe that the range of x depends on n (ie $c = c(n)$) and the domain of ψ $(D(\psi))$ is a superset of the domain of φ $\left(D(\varphi)\right)$ $\left(\text{i.e } D(\varphi) \subset D(\psi)\right)$. φ and ψ are called the weight or scale function for geometric distribution.
In this research work, we seek for such $\psi(n,q)$ that will modify equation (1.5).
Now, let

$$\psi(n,q) = (1-q)q^{n};\ \forall\ \ n \in \mathbb{N} \cup \{0\}. \qquad (1.7)$$

Using (1.6) and (1.7) we have that $\forall\ \ n \in Z$,

$$f(x;n,q) = (1-q)q^{x-c(n)};\ \ \ x \geq c(n) \qquad (1.8)$$

Where, $c(-n) = n + 1$ or $c(n) = -n + 1$ depending on wether the first success occurs atleast n-steps before the xth trial or atmost n-steps after the xth trial. Equation (1.8) is called the $c(n)$-geometric (simple n-geometric distribution if $n \geq 0$ and reverse n-geometric distribution if $n \leq 0$) distribution.

Definition 1.1

For a given $n \in \mathbb{N} \cup \{0\}$, a random variable X is said to have c(n)-geometric distribution if the probability mass function (pmf) is given by

$$f(x; n, q) = (1-q)q^{x-c(n)}; \ \ x \geq c(n)$$

And the cumulative distribution function (cdf) is given by

$$F(x; n) = \begin{cases} 0 & \text{if } x < c(n) \\ 1 - q^{x-c(n)+1} & \text{if } c(n) \leq x < \infty \\ 1 & \text{if } x \geq \infty \end{cases} \quad (1.9)$$

We shall state results for equation (1.8) when $c(n) = -n + 1$, called the n-geometric distribution since the reverse case can easily be deduced by replacing $-n$ by n. Hence $\forall \ n \in \mathbb{N} \cup \{0\}$ we consider;

$$f(x; n, q) = (1-q)q^{x+n-1}; \quad x \geq -n+1 \quad (1.10)$$

Observe that if $n = 0$, (1.10) reduces to (1.1) and if $n = 1$, (1.10) reduces to (1.2). The random variable X that defines (1.1) and (1.2) are often described as; the number of independent (Bernoulli's) trials on which the first success occurs and the number of failures before the first success occurs. These corresponds to the 0 (zero)-geometric distribution, which implies that the first success occurs at the xth trial and 1 (one)-geometric distribution which implies that the first success occurs at the (x+1)th trial. Hence, in general, the random variable X that describes (counts) the number of independent trials on which the first success occurs at the (next) nth additional trials is defined by equation (1.10).

2.0 MAIN RESULTS

Theorem 2.1

Let X be a random variable that has the pmf in (1.8) then the cdf is given by (2.2).

Proof:

Since X is given by (1.8), we shall have:

$$F(x) = \sum_{y=c(n)}^{x} f(y) = \sum_{y=c(n)}^{x} (1-q)q^{y-c(n)}$$

$$= (1-q)\sum_{y=c(n)}^{x} q^{y-c(n)} = 1 - q^{x-c(n)+1} \quad (2.1)$$

Now, if $x \geq \infty$, we will have

$$\lim_{x\to\infty} F(x) = 1 \quad (2.2)$$

This completes the proof and consequently f is a well-defined pmf.

Theorem2.3

If X is a random variable with the probability mass function in (1.10) then the mean $E(X; n)$, variance $Var(X; n)$, probability generating function $E(t^x; n)$, moment generating function $E(e^t; n)$ and the rth factorial moment $E\left(X^{(r)}; n\right)$ are given by:

i. $E(X;n) = \frac{1-n(1-q)}{1-q}$

ii. $Var(X;n) = \frac{q}{(1-q)^2}$

iii. $E(t^x;n) = \frac{(1-q)t^{1-n}}{(1-qt)}$

iv. $E(e^t;n) = \frac{(1-q)e^{t(1-n)}}{(1-qe^t)}$

v. $E\left(X^{(r)};n\right) = \left(\frac{q}{1-q}\right)^r \sum_{s=0}^{r} \binom{1-n}{s} \left(\frac{1-q}{q}\right)^s;\ x^{(r)} = x(x-1)\ldots(x-r+1)$

Proof:

(i)

$E(x) = \sum_{R_x} xf(x) = (1-q)q^{n-1} \sum_{x\geq -n+1} xq^x.$

$$= (1-q)q^{n-1}S(n) \quad (2.3)$$

where, $S(n) = \sum_{x\geq -n+1} xq^x$, and we define $S_r(n) := \sum_{x=-n+1}^{r} xq^x$ So that

$$S_r(n) = \frac{(-n+1)q^{-n+1}}{(1-q)} + \frac{q^{-n+2}(1-q^{r+n-1})}{(1-q)^2} - \frac{rq^{r+1}}{(1-q)} \quad (2.4)$$

Consequently,

$$S(n) = \frac{(1-n)q^{-n+1}}{(1-q)} + \frac{q^{-n+2}}{(1-q)^2} \quad (2.5)$$

Using (2.3) and (2.5), we have

$$E(x) = (1-q)q^{-n+2}\left(\frac{(-n+1)q^{-n+1}}{(1-q)} + \frac{q^{-n+2}}{(1-q)^2}\right) = \frac{1-n(1-q)}{1-q} \quad (2.6)$$

(ii)

$$E(x^2) = (1-q)q^{n-1} \sum_{x\geq -n+1} x^2 q \quad = (1-q)q^{n-1}T(n) \quad (2.7)$$

where $T(n) = \sum_{x\geq -n+1} x^2 q^x$ and we define $T_r(n) := \sum_{x=-n+1}^{r} x^2 q^x$ so that we have

$$T_r(n) = \frac{(-n+1)^2 q^{-n+1}}{1-q} + U_r(n) - \frac{r^2 q^{r+1}}{1-q} \quad (2.8)$$

where $U_r(n) = \sum_{x=2}^{r+n} \frac{2(x-n-\frac{1}{2})q^{-n+x}}{1-q}$, Consequently, we have

$$U_r(n) = \frac{(-2n+3)q^{-n+2}}{(1-q)^2} + \frac{2q^{-n+3}(1-q^{r+n-2})}{(1-q)^3} - \frac{(2r-1)q^{r+1}}{(1-q)^2} \qquad (2.9)$$

By (2.8) and (2.9) we have that

$$T_r(n) = \frac{(-n+1)^2q^{-n+1}}{(1-q)} + \frac{(-2n+3)q^{-n+2}}{(1-q)^2} + \frac{2q^{-n+3}(1-q^{r+n-2})}{(1-q)^3} - \frac{(2r-1)q^{r+1}}{(1-q)^2} - \frac{r^2q^{r+1}}{(1-q)} \quad (2.10)$$

Thus,

$$T(n) = \frac{(-n+1)^2q^{-n+1}}{(1-q)} + \frac{(-2n+3)q^{-n+2}}{(1-q)^2} + \frac{2q^{-n+3}}{(1-q)^3} \qquad (2.11)$$

Substituting (2.11) into (2.7) we have that

$$E(x^2) = (1-q)q^{n-1}\left(\frac{(-n+1)^2q^{-n+1}}{(1-q)} + \frac{(-2n+3)q^{-n+2}}{(1-q)^2} + \frac{2q^{-n+3}}{(1-q)^3}\right)$$

$$= \frac{(n-1)^2+(1-2n(n-1))q+n^2q^2}{(1-q)^2} \qquad (2.12)$$

Hence,

$$Var(x) = E(x^2) - (E(x))^2$$

$$= \frac{(n-1)^2+(1-2n(n-1))q+n^2q^2}{(1-q)^2} - \left(\frac{(1-n)+nq}{1-q}\right)^2 = \frac{q}{(1-q)^2}$$

(iii) and **(iv)**

Let $m = m(t)$ be continuous real-valued function, suppose there exists $t^* \in \mathbb{R}$ which is a zero of $m(t) - 1$, then we compute:

$$E(m^x) = \sum_{x\geq -n+1} m^x(1-q)q^{x+n-1}$$

$$= (1-q)q^{n-1}\sum_{x\geq -n+1}(mq)^x = \frac{(1-q)m^{1-n}}{1-qm}$$

Thus if we take $m(t) = t$ (as in (iii)) and $m(t) = e^t$ (as in (iv)), then the results follows.

(v) $E(X^{(r)};n) = \sum_{x\geq -n+1} x^{(r)}(1-q)q^{x+n-1} = (1-q)q^{n-1}\sum_{x\geq -n+1} x^{(r)}q^{x}$

$$= (1-q)q^{n+r-1}\sum_{x\geq -n+1}\frac{d^r}{dq^r}(q^x)$$

By uniform convergence of the series $\sum_{x\geq -n+1} q^x$ and continuity of the derivative $\frac{d^r}{dq^r}$, we have that

$$E(X^{(r)};n) = (1-q)q^{n+r-1}\frac{d^r}{dq^r}\left(\frac{q^{-n+1}}{1-q}\right)$$

$$= (1-q)q^{n+r-1}\sum_{s=0}^{r}\binom{1-n}{s}q^{1-n-s}(1-q)^{-(r-s+1)}$$

$$= \left(\frac{q}{1-q}\right)^r\sum_{s=0}^{r}\binom{1-n}{s}\left(\frac{1-q}{q}\right)^s$$

Theorem2.4

If X is a random variable with the pmf in (2.1) and $\eta(t;n,q) \coloneqq E(t^x)$, then for any integer $r \geq 0$;

i. $\frac{d^r\eta}{dt^r} = r!\,(1-q)\sum_{j=1}^{r+1}(-1)^{r-j+1}\binom{n+r-1-j}{n-2}a^{n+r-j}b^jq^{j-1}$

ii. $\left.\frac{d^r\eta}{dt^r}\right|_{t=1} = r!\sum_{j=1}^{r+1}(-1)^{r-j+1}\binom{n+r-1-j}{n-2}\left(\frac{q}{1-q}\right)^{j-1}$

Where $a = a(t) = t^{-1}$ and $b = b(t;q) = (1-tq)^{-1}$.

Proof:

i. $\eta(t) = (1-q)t^{-n+1}(1-tq)^{-1} = (1-q)g(t)$,

where $g(t) = t^{-n+1}(1-tq)^{-1}$, $g^{(r)} = \frac{d^rg}{dt^r}$

$$g^{(1)}(t) = (1-n)t^{-n}(1-tq)^{-1} + t^{1-n}(1-tq)^{-2}q$$

$$g^{(2)}(t) = (1-n)(-n)t^{-(n+1)}(1-tq)^{-1} + (1-n)t^{-n}(1-tq)^{-2}q + (1-n)t^{-n}(1-tq)^{-2}q + t^{1-n}(1-tq)^{-3}\,q2q$$

$g^{(3)}(t) = (1-n)(-n)t^{-(n+2)}(1-tq)^{-1} + (1-n)(-n)t^{-(n+1)}(1-tq)^{-2}q + (1-n)(-n)t^{-(n+1)}(1-tq)^{-2}q + (1-n)t^{-n}(1-tq)^{-3}q2q + (1-n)(-n)t^{-(n+1)}(1-tq)^{-2}q + (1-n)t^{-n}(1-tq)^{-3}q2q + (1-n)t^{-n}(1-tq)^{-3}q2q + t^{1-n}(1-tq)^{-4}q2q3q.$

If we continue in this manner, after some algebraic simplification, one can easily deduce for any integer $r \geq 1$;

$$g^{(r)}(t) = \sum_{j=1}^{r+1} (-1)^{r+1-j} \binom{r}{j-1} \frac{(n+r-1-j)!}{(n-2)!} a^{n+r-j} b^j q^{j-1} (j-1)!$$

So that:

$$\eta^{(r)}(t) = r!\,(1-q) \sum_{j=1}^{r+1} (-1)^{r-j+1} \binom{n+r-1-j}{n-2} a^{n+r-j} b^j q^{j-1}$$

Alternatively, one can show by induction that $\eta^{(r)}(t)$ holds for every $r \geq 1$. To do this, we state the following claims which will be used in the sequel.

Claim 1.

$$(2-m-n)\binom{1-n}{m-1} + (k+1-m)\binom{1-n}{m} = (k+1)\binom{1-n}{m}.$$

Proof

$$\begin{aligned}(2-m-n)\binom{1-n}{m-1} &+ (k+1-m)\binom{1-n}{m} \\ &= \frac{(1-n)!}{(1-m-n)!\,(m-1)!} + \frac{(k+1-m)(1-n)!}{(m)!\,(1-m-n)!} = \frac{(k+1)(1-n)!}{(1-m-n)!\,(m)!} \\ &= (k+1)\binom{1-n}{m}\end{aligned}$$

This completes the proof.

Claim 2.

$$(-1)^{r+1-j}\binom{n+r-1-j}{n-2} = \binom{-n+1}{r-j+1}$$

Proof

$$(-1)^{r+1-j}\binom{n+r-1-j}{n-2} = (-1)^{r+1-j}\binom{n+r-1-j}{r+1-j}$$

$$= (-1)^{r-j+1}\binom{r-j+1+n-2}{r-j+1} = \binom{-n+1}{r-j+1}.$$

This completes the proof.

Thus, by claim 2 we have

$$\frac{d^r\eta}{dt^r} = r!\,(1-q)\sum_{j=1}^{r+1}\binom{1-n}{r-j+1} a^{n+r-j}\, b^j\, q^{j-1}$$

$$= r!\,(1-q)\sum_{m=0}^{r}\binom{1-n}{m} a^{n+m-1}\, b^{r-m+1}\, q^{r-m}$$

Now, if $r = 1$, we have

$$\frac{d\eta}{dt} = (1-q)\left[\binom{1-n}{0} a^{n-1}\, b^2\, q^1 + \binom{1-n}{1} a^n\, b\right]$$

$$= (1-q)[t^{1-n}(1-tq)^{-2}\, q + (1-n)t^{-n}\, b(1-tq)^{-1}]$$

which is true .

Suppose it is true for $r = k$, we now show that for $r = k+1$, $\frac{d^{k+1}\eta}{dt^{k+1}}$ remains true.

$$\frac{d^{k+1}\eta}{dt^{k+1}} = \frac{d}{dt}\left(\frac{d^k\eta}{dt^k}\right) = k!\,(1-q)\sum_{m=0}^{k}\binom{1-n}{m}\frac{d}{dt}\{a^{m+n-1}b^{k-m+1}\}q \qquad (2.13)$$

But,

$$\frac{d}{dt}\{a^{m+n-1}b^{k-m+1}\} = -(m-k-1)t^{-(m+n-1)}(1-tq)^{-(k-m+2)}q$$

$$+(1-m-n)t^{-(m+n)}(1-tq)^{-(k-m+1)}$$

$$= (k+1-m)\, a^{(m+n-1)}\, b^{(k+1)-m+1}\, q + (1-m-n)\, a^{m+n}\, b^{k+1-m} \qquad (2.14)$$

Substituting equation (2.14) into equation (2.13), we have that

$$\frac{d^{k+1}\eta}{dt^{k+1}} =$$

$$k!\,(1-q)\left[\sum_{m=0}^{k}(k+1-m)\binom{1-n}{m}a^{m+n-1}\,b^{(k+1)-m+1}\,q^{k-m+1} + \sum_{m=0}^{k}(1-m-n)\binom{1-n}{m}a^{m+n}\,b^{k+1-m}\,q^{k-m}\right]$$

$$= k!\,(1-q)\left[(k+1)\,a^{n-1}\,b^{(k+1)+1}\,q^{k+1} + \sum_{m=0}^{k}(k+1-m)\binom{1-n}{m}a^{m+n-1}b^{(k+1)-m+1}q^{k+1-m} + (1-k-n)a^{k+n}\,b\binom{1-n}{k} + \sum_{m=0}^{k-1}(1-m-n)\binom{1-n}{m}a^{m+n}\,b^{k+1-m}\,q^{k-m}\right]$$

$$= (1-q)(k+1)!\;a^{n-1}\,b^{(k+1)+1}\,q^{(k+1)}$$

$$+(1-q)\sum_{m=0}^{k-1}k!\,(1-m-n)\binom{1-n}{m}a^{m+n}b^{k+1-m}q^{k-m}$$

$$+(1-q)\sum_{m=0}^{k}k!\,(k+1-m)\binom{1-n}{m}a^{m+n-1}b^{(k+1)-m+1}q^{(k+1)-m} + (1-q)(k+1)!\;a^{n-1}\,b^{(k+1)+1}\,q^{k+1}$$

$$= (1-q)(k+1)!\;a^{n-1}\,b^{(k+1)+1}\,q^{(k+1)} + (1-q)k!\sum_{m=0}^{k}(2-m-n)\binom{1-n}{m}a^{m+n-1}\,b^{(k+1)-m+1}q^{k+1-m}$$

$$+(1-q)\textstyle\sum_{m=0}^{k}k!\,(k+1-m)\binom{1-n}{m}a^{m+n-1}b^{(k+1)-m+1}q^{(k+1)-m}$$

$$+ (1-q)(k+1)!\;a^{n-1}\,b^{(k+1)+1}\,q^{k+1}.$$

$$= (1-q)(k+1)!\;a^{n-1}\,b^{(k+1)+1}\,q^{(k+1)}$$

$$+(1-q)k!\sum_{m=1}^{k}\left[(2-m-n)\binom{1-n}{m-1}+(k+1-m)\binom{1-n}{m}\right]$$

$$+a^{m+n-1}\,b^{(k+1)-m+1}q^{(k+1)-m}+(1-q)(k+$$
$1)!\ a^{n-1}\,b^{k+2}\,q^{k+1}.$

Using claim 1 we have

$$(1-q)(k+1)!\ a^{n-1}\,b^{(k+1)+1}\,q^{(k+1)}$$
$$+(1-q)\sum_{m=1}^{k}(k+1)!\binom{1-n}{m}a^{m+n-1}\,b^{(k+1)-m+1}q^{k+1-m}$$
$$+(1-q)(k+1)!\ a^{n-1}\,b^{(k+1)+1}\,q^{k+1}.$$

$$=(1-q)(k+1)!\sum_{m=0}^{k+1}\binom{1-n}{m}a^{m+n-1}\,b^{k-m+2}q^{k-m+1}$$

This completes the proof.
(ii) Since when $t=1$, we have $a=1$ and $b=(1-\mathrm{q})^{-1}$ so that the result follows immediately.
Theorem2.5
If X is a random variable with the *pmf* in (2.1) and $M(t\colon n,q)=\ E(e^{tx})$, then for any integer $r\geq 1$;

(i) $\frac{d^rM}{dt^r}=(1-q)\sum_{k=1}^{r+1}(-1)^{r-k+1}a_n^{(\sum_{i=1}^{k}j_i)}b_{(t)}^{n-k}c_{(t,q)}^{k}q^{k-1}(k-1)!$, if $\sum_{i=1}^{k}j_i=r-k+1$

(ii) $\left.\frac{\mathrm{d^rM}}{\mathrm{dt^r}}\right|_{t=0}=\sum_{k=1}^{r+1}(-1)^{r-k+1}a_n^{(\sum_{i=1}^{k}j_i)}\left(\frac{q}{1-q}\right)^{k-1}(k-1)!$, if $\sum_{i=1}^{k}j_i=r-k+1$

where, $a_n^{j_i}=\ (n-i)^{j_i},\ b_{(t)}=e^{-t}, and\ c_{(t,q)}=(1-qe^t)^{-1}$ and the summation is taken over all integer $j_i\ (i=1,2,\dots,k)$ satisfying the linear diaphaqntine equation $\sum_{i=1}^{k}j_i=r-k+1.$

Proof:

(i) $M(t)=(1-q)\,h(t)\colon\ h(t)=e^{t(1-n)}(1-qe^t)^{-1}$

$h^{(1)}=(1-n)e^{t(1-n)}(1-qe^t)^{-1}+e^{t(2-n)}(1-qe^t)^{-2}q,$

$$h^{(2)} = (1-n)^2 e^{t(1-n)}(1-qe^t)^{-1} + (1-n)e^{t(2-n)}(1-qe^t)^{-2}q + (2-n)e^{t(2-n)}(1-qe^t)^{-2}q$$

$$+ e^{t(3-n)}(1-qe^t)^{-3}q2q.$$

$$h^{(3)} = (1-n)^3 e^{t(1-n)}(1-qe^t)^{-1} + (1-n)^2 e^{t(2-n)}(1-qe^t)^{-2}q$$

$$+(1-n)(2-n)e^{t(2-n)}(1-qe^t)^{-2}q + (1-n)e^{t(3-n)}(1-qe^t)^{-3}q\ 2q$$

$$+(2-n)^2 e^{t(2-n)}(1-qe^t)^{-2}q + (2-n)e^{t(3-n)}(1-qe^t)^{-3}q2q$$

$$+(3-n)e^{t(3-n)}(1-qe^t)^{-3}q\ 2q + e^{t(4-n)}(1-qe^t)^{-4}q\ 2q\ 3q.$$

If we continue in this manner for any integer $r \geq 1$, and define

$a_n^{j_i} = (n-i)^{j_i},\ b(t) = e^{-t}, and\ c(t,q) = (1-qe^t)^{-1}$, after some algebraic simplifications, deductively we will have

$$h^{(r)} = \sum_{k=1}^{r+1}(-1)^{r-k+1} a_n^{(\sum_{i=1}^{k} j_i)} b_{(t)}^{n-k} c_{(t,q)}^{k} q^{k-1}(k-1)!;\ \text{if}\ \sum_{i=1}^{k} j_i = r-k+1$$

The inductive prove follows the same manner as in theorem 2.4 and this establishes the result.
(ii) As a consequence of (i) above, if $t = 0$, then $b(0) = 1\ and\ c(0,q) = (1-q)^{-1}$, hence the result follows.

CONCLUSION

We conclude this work by stating that, in theorem 2.2 and theorem 2.3, if we take $n = 0$ or $n = 1$, we obtain the results of the standard (classical or usual) geometric distribution. The results of theorem 2.4, theorem 2.4 and theorem 2.5 are entirely new. We also remark that the results obtained by some of the above mentioned authors can further be generalized in the direction of this research work, which is a follow up to this paper in the series of our research work. That is by assuming the consecutive k success occurs atleast n-steps before the xth trial or atmost n-steps after the xth trial, so that if $n = 0$, we recover the results of consecutive k success that occurs at the xth trial. Furthermore, if we represent this occurrence of consecutive k success as target point on the random variable X, then the target set T is a point set (singleton) i.e. $T = \{x\}$. While based on our assumption, for every n ∈ Z, $T_n = \{x-n, x-n+1, \ldots, x, x+1, \ldots, x+n\}$ ($T_{n\leq 0} = \{x-n, x-n+1, \ldots, x\}$ and $T_{n\geq 0} = \{x, x+1, \ldots, x+n\}$) so that one can easily see that $T_0 = T$.

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